# Calculus AB
Lesson: Thursday, April 9

**Learning Target:**

Students will integrate functions involving inverse trig functions.

**Let's Get Started:**

Read Article: Integrals Involving Inverse Trig Functions

Watch Video: Integrals Involving Inverse Trig

# Practice:

1. Here are the formulas we will be using for this lesson:

**Integrals Involving the Inverse Trig Functions**

$$\int \frac{du}{\sqrt{1-u^2}} = \arcsin u + C \qquad \int \frac{du}{\sqrt{a^2-u^2}} = \arcsin \frac{u}{a} + C$$

$$\int \frac{du}{1+u^2} = \arctan u + C \qquad \int \frac{du}{a^2+u^2} = \frac{1}{a}\arctan \frac{u}{a} + C$$

$$\int \frac{du}{u\sqrt{u^2-1}} = \operatorname{arcsec} |u| + C \qquad \int \frac{du}{u\sqrt{u^2-a^2}} = \frac{1}{a}\operatorname{arcsec} \frac{|u|}{a} + C$$

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2. We will use these formulas to solve each problem in this lesson.

3. These formulas was used to complete the following problem:

$$\int \frac{dx}{\sqrt{49-x^2}} = \int \frac{dx}{\sqrt{(7)^2-x^2}}$$

$$= \sin^{-1}\left(\frac{x}{7}\right) + K$$

We could also write this answer as:

$$\arcsin\left(\frac{x}{7}\right) + K$$

Here is another worked out examples:

$$\int_0^1 \frac{2\,dx}{\sqrt{9-4x^2}}$$

Let $u = 2x$ then $du = 2\ dx$.

Our integral becomes:

$$\int_0^1 \frac{2\,dx}{\sqrt{9-4x^2}} = \int_{x=0}^{x=1} \frac{du}{\sqrt{(3)^2 - u^2}}$$

$$= \left[\sin^{-1}\left(\frac{u}{3}\right)\right]_{x=0}^{x=1}$$

$$= \left[\sin^{-1}\left(\frac{2x}{3}\right)\right]_0^1$$

$$= \left[\sin^{-1}\left(\frac{2}{3}\right) - \sin^{-1}0\right]$$

$$= 0.7297$$

We could have done it in much fewer steps by leaving it in terms of "$u$", as follows:

$$\int_0^1 \frac{2\,dx}{\sqrt{9-4x^2}} = \int_0^2 \frac{du}{\sqrt{(3)^2 - u^2}}$$

$$= \left[\sin^{-1}\left(\frac{2}{3}\right) - \sin^{-1}0\right]$$

$$= 0.7297$$

Note the change in the limits when $dx$ is changed to $du$ during the integration.

This is because we let $u = 2x$.

Hence the limits for $x$ of

$x = 0$ and $x = 1$

have to be changed for $u$ to

$u = 0$ and $u = 2$.

One more worked out example:

Find the area bounded by the curve $y = \dfrac{1}{1+x^2}$ and the lines $x = 0$, $y = 0$ and $x = 2$.

The curve $y = \dfrac{1}{1+x^2}$ lies entirely above the $x$-axis for all values of $x$, so to find the area we can simply integrate. (If part of the curve was below the $x$-axis, we would need to split it into different portions and take absolute values.)

$$
\begin{aligned}
\text{Area} &= \int_0^2 \frac{dx}{1+x^2} \\
&= \left[\tan^{-1} x\right]_0^2 \\
&= \left[\tan^{-1} 2 - \tan^{-1} 0\right] \\
&= 1.1071 \text{ units}^2
\end{aligned}
$$

**Practice: Evaluate the following.**

1. $\displaystyle\int \frac{3\ dx}{25 + 16x^2}$

2. $\displaystyle\int \frac{2\ dx}{x^2 + 8x + 17}$

3. $\displaystyle\int \frac{dx}{\sqrt{2x - x^2}}$

# Answer Key:

Once you have completed the problem, check your answers here.

1. We can write our integral as:

$$\int \frac{3\ dx}{25+16x^2} = 3\int \frac{dx}{(5)^2+(4x)^2}$$

For the formula, we need:

$a = 5$; $u = 4x$; $du = 4\ dx$.

Re-arranging that last expression gives: $\frac{du}{4} = dx$.

We are now ready to perform the integration.

$$\int \frac{3\ dx}{25+16x^2} = \frac{3}{4}\int \frac{du}{(5)^2+u^2}$$

$$= \frac{3}{4}\times\frac{1}{5}\tan^{-1}\left(\frac{u}{5}\right)+K$$

$$= \frac{3}{20}\tan^{-1}\left(\frac{4x}{5}\right)+K$$

2. $$\int \frac{2\,dx}{x^2+8x+17}$$

Now

$$x^2+8x+17 = \left(x^2+8x+16\right)+1$$

$$= (x+4)^2+1$$

So if we let $u = x+4$, then $du = dx$ and we have:

$$\int \frac{2\ dx}{x^2+8x+17} = 2\int \frac{du}{u^2+1}$$

$$= 2\tan^{-1}u+K$$

$$= 2\tan^{-1}(x+4)+K$$

# Answer Key:

Once you have completed the problem, check your answers here.

3. $$\int \frac{dx}{\sqrt{2x - x^2}}$$

This is not in the form of either of our new formulas, but we can do some juggling to get it into a useful form.

First, we recognise that

$$2x - x^2 = -(x^2 - 2x)$$

We now add 1 at the front, then compensate for it inside the bracket:

$$= 1 - (x^2 - 2x + 1)$$

$$= 1 - (x - 1)^2$$

With $a = 1$; $u = x - 1$, and $du = dx$, our integral becomes:

$$\int \frac{dx}{\sqrt{2x - x^2}} = \int \frac{du}{\sqrt{1 - u^2}}$$

$$= \arcsin u + K$$

$$= \arcsin (x - 1) + K$$

# Additional Practice:

[Additional Practice with Answers]

In your Calculus book Read through Section 5.7 and complete problems 1, 9, 23, 33, 39 on page 385